

ZERO-DIMENSIONAL SYMPLECTIC ISOLATED COMPLETE INTERSECTION SINGULARITIES

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ABSTRACT. We study the local symplectic algebra of the 0-dimensional isolated complete intersection singularities. We use the method of algebraic restrictions to classify these symplectic singularities. We show that there are non-trivial symplectic invariants in this classification.

1. INTRODUCTION

The problem of symplectic classification of singular varieties was introduced by V. I. Arnold in [A1]. Arnold showed that the A_{2k} singularity of a planar curve (the orbit with respect to the standard \mathcal{A} -equivalence of parameterized curves) split into exactly $2k + 1$ symplectic singularities (orbits with respect to the symplectic equivalence of parameterized curves). He distinguished different symplectic singularities by different orders of tangency of the parameterized curve to the *nearest* smooth Lagrangian submanifold. Arnold posed a problem of expressing these new symplectic invariants in terms of the local algebra's interaction with the symplectic structure and he proposed to call this interaction the **local symplectic algebra**. This problem was studied by many authors mainly in the case of singular curves.

In [IJ1] G. Ishikawa and S. Janeczko classified symplectic singularities of curves in the 2-dimensional symplectic space. A symplectic form on a 2-dimensional manifold is a special case of a volume form on a smooth manifold. The generalization of results in [IJ1] to volume-preserving classification of singular varieties and maps in arbitrary dimensions was obtained in [DR]. The orbit of the action of all diffeomorphism-germs agrees with the volume-preserving orbit in the \mathbb{C} -analytic category for germs which satisfy a special weak form of quasi-homogeneity e.g. the weak quasi-homogeneity of varieties is a quasi-homogeneity with non-negative weights $\lambda_i \geq 0$ and $\sum_i \lambda_i > 0$.

P. A. Kolgushkin classified stably simple symplectic singularities of parameterized curves in the \mathbb{C} -analytic category ([K]).

In [DJZ2] the local symplectic algebra of singular quasi-homogeneous subsets of a symplectic space was explained by the algebraic restrictions of the symplectic form to these subsets. The generalization of the Darboux-Givental theorem ([AG]) to germs of arbitrary subsets of the symplectic space obtained in [DJZ2] reduces the problem of symplectic classification of germs of quasi-homogeneous subsets to the problem of classification of algebraic restrictions of symplectic forms to these

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subsets. For non-quasi-homogeneous subsets there is one more cohomological invariant apart of the algebraic restriction ([DJZ2], [DJZ1]). The method of algebraic restrictions is a very powerful to study the problem of local symplectic algebra of 1-dimensional singular analytic varieties since the space of algebraic restrictions of closed 2-forms to a 1-dimensional singular analytic variety is finite-dimensional ([D]). By this method complete symplectic classifications of the $A-D-E$ singularities of planar curves and S_5 singularity were obtained in [DJZ2]. These results were generalized to other 1-dimensional isolated complete intersection singularities: the S_μ symplectic singularities for $\mu > 5$ in [DT1], the $T_7 - T_8$ symplectic singularities in [DT2] and the $W_8 - W_9$ symplectic singularities in [T].

In this paper we show that some non-trivial symplectic invariants appear not only in the case of singular curves but also in the case of multiple points. We consider the symplectic classification of the 0-dimensional isolated complete intersection singularities (ICISs) in the symplectic space $(\mathbb{C}^{2n}, \omega)$. We need to introduce a symplectic V -equivalence to study this problem since the ideals of function-germs that we consider do not have the property of zeros.

We recall that ω is a \mathbb{C} -analytic symplectic form on \mathbb{C}^{2n} if ω is a \mathbb{C} -analytic nondegenerate closed 2-form, and $\Phi : \mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n}$ is a symplectomorphism if Φ is a \mathbb{C} -analytic diffeomorphism and $\Phi^*\omega = \omega$.

Definition 1.1. Let $f, g : (\mathbb{C}^{2n}, 0) \rightarrow (\mathbb{C}^k, 0)$ be \mathbb{C} -analytic map-germs on the symplectic space $(\mathbb{C}^{2n}, \omega)$. f, g are **symplectically V -equivalent** if there exists a symplectomorphism-germ $\Phi : (\mathbb{C}^{2n}, 0, \omega) \rightarrow (\mathbb{C}^{2n}, 0, \omega)$ and a \mathbb{C} -analytic map-germ $M : (\mathbb{C}^{2n}, 0) \rightarrow GL(k, \mathbb{C})$ such that $f \circ \Phi = M \cdot g$.

If $\Phi : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^m, 0)$ is a \mathbb{C} -analytic map-germ then for an ideal I in the ring of \mathbb{C} -analytic function-germs on \mathbb{C}^m we denote by Φ^*I the following ideal of $\{f \circ \Phi : f \in I\}$ in the ring of \mathbb{C} -analytic function-germs on \mathbb{C}^n . The (symplectic) V -equivalence of map-germs $f = (f_1, \dots, f_k), g = (g_1, \dots, g_k) : (\mathbb{C}^{2n}, 0) \rightarrow (\mathbb{C}^k, 0)$ corresponds to the following (symplectic) equivalence of finitely-generated ideals $\langle f_1, \dots, f_k \rangle$ and $\langle g_1, \dots, g_k \rangle$ (see [AVG]).

Definition 1.2. Ideals $\langle f_1, \dots, f_k \rangle$ and $\langle g_1, \dots, g_k \rangle$ of \mathbb{C} -analytic function-germs at 0 on the symplectic space $(\mathbb{C}^{2n}, \omega)$ are **symplectically equivalent** if there exists a symplectomorphism-germ $\Phi : (\mathbb{C}^{2n}, 0, \omega) \rightarrow (\mathbb{C}^{2n}, 0, \omega)$ such that $\Phi^* \langle f_1, \dots, f_k \rangle = \langle g_1, \dots, g_k \rangle$.

In this paper we present the complete symplectic classification of the $I_{a,b}, I_{2a+1}, I_{2a+4}, I_{a+5}, I_{10}^*$ singularities. For $n = 1$ all V -orbits coincide with symplectic V -orbits. The situation for $n \geq 2$ is different: the $I_{a,b}$ singularities split into two symplectic V -orbits, the $I_{2a+1}, I_{2a+4}, I_{a+5}$ singularities split into three symplectic orbits and finally I_{10}^* singularity splits into four symplectic V -orbits. The symplectic V -orbits of a map $f = (f_1, \dots, f_{2n})$ are distinguished by the order of vanishing of a pullback of the germ of the symplectic form to a \mathbb{C} -analytic non-singular submanifold M of the minimal dimension such that the ideal of \mathbb{C} -analytic function-germs vanishing on M is contained in the ideal $\langle f_1, \dots, f_{2n} \rangle$ (see Definition 3.2).

To obtain these results we need some reformulation and modification of the method of algebraic restrictions. We present it in Section 2. In Section 3 we give the definitions of discrete symplectic invariants which completely distinguish symplectic V -singularities considered in this paper. We recall basic facts on the

classification of V -simple maps in Section 4. In Section 5 we prove the symplectic V -classification theorem for 0-dimensional ICISs (Theorem 5.1).

2. THE METHOD OF ALGEBRAIC RESTRICTIONS FOR THE SYMPLECTIC V -EQUIVALENCE.

In this section we present basic facts on the method of algebraic restrictions adapted to the case of the symplectic V -equivalence. The proofs of all results are small modifications of the proofs of analogous results in [DJZ2].

Given a germ at 0 of a non-singular \mathbb{C} -analytic submanifold M of \mathbb{C}^m denote by $\Lambda^p(M)$ the space of all germs at 0 of \mathbb{C} -analytic differential p -forms on M . By $\mathcal{O}(M)$ denote the ring of \mathbb{C} -analytic function-germs on M at 0. Given an ideal I in $\mathcal{O}(M)$ introduce the following subspace of $\Lambda^p(M)$:

$$\mathcal{A}_0^p(I, M) = \{\alpha + d\beta : \alpha \in I\Lambda^p(M), \beta \in I\Lambda^{p-1}(M)\}.$$

The relation $\omega \in I\Lambda^p(M)$ means that $\omega = \sum_{i=1}^k f_i \alpha_i$, where $\alpha_i \in \Lambda^p(M)$ and $f_i \in I$ for $i = 1, \dots, k$.

Definition 2.1. Let I be an ideal of $\mathcal{O}(M)$ and let $\omega \in \Lambda^p(M)$. The **algebraic restriction** of ω to I is the equivalence class of ω in $\Lambda^p(M)$, where the equivalence is as follows: ω is equivalent to $\tilde{\omega}$ if $\omega - \tilde{\omega} \in \mathcal{A}_0^p(I, M)$.

Notation. The algebraic restriction of the germ of a p -form ω on M to the ideal I in $\mathcal{O}(M)$ will be denoted by $[\omega]_I$. Writing $[\omega]_I = 0$ (or saying that ω has zero algebraic restriction to I) we mean that $[\omega]_I = [0]_I$, i.e. $\omega \in \mathcal{A}_0^p(I, M)$.

Definition 2.2. Two algebraic restrictions $[\omega]_I$ and $[\tilde{\omega}]_{\tilde{I}}$ are called **diffeomorphic** if there exists the germ of a diffeomorphism $\Phi : M \rightarrow \tilde{M}$ such that $\Phi^*(\tilde{I}) = I$ and $[\Phi^*\tilde{\omega}]_I = [\omega]_I$.

Definition 2.3. The germ of a function, a differential k -form, or a vector field α on $(\mathbb{C}^m, 0)$ is **quasi-homogeneous** in a coordinate system (x_1, \dots, x_m) on $(\mathbb{C}^m, 0)$ with positive integer weights $(\lambda_1, \dots, \lambda_m)$ if $\mathcal{L}_E \alpha = \delta \alpha$, where $E = \sum_{i=1}^m \lambda_i x_i \frac{\partial}{\partial x_i}$ is the germ of the **Euler vector field** on $(\mathbb{C}^m, 0)$ and δ is called the quasi-degree.

It is easy to show that α is quasi-homogeneous in a coordinate system (x_1, \dots, x_m) with weights $(\lambda_1, \dots, \lambda_m)$ if and only if $F_t^* \alpha = t^\delta \alpha$, where

$$(2.1) \quad F_t(x_1, \dots, x_m) = (t^{\lambda_1} x_1, \dots, t^{\lambda_m} x_m).$$

Definition 2.4. A finitely generated ideal I of $\mathcal{O}(\mathbb{C}^m)$ is **quasi-homogeneous** if there exist generators of I which are quasi-homogeneous in the same coordinate system (x_1, \dots, x_m) on \mathbb{C}^m with the same positive integer weights $(\lambda_1, \dots, \lambda_m)$.

To prove the generalization of Darboux-Givental theorem suitable for the symplectic V -equivalence of maps or the symplectic equivalence of ideals of function-germs we need the following version of the Relative Poincare Lemma.

Lemma 2.5. *Let I be a finitely generated quasi-homogeneous ideal in $\mathcal{O}(\mathbb{C}^m)$. If $\omega \in I\Lambda^p(\mathbb{C}^m)$ is closed then there exists $\alpha \in I\Lambda^{p-1}(\mathbb{C}^m)$ such that $\omega = d\alpha$.*

Proof. We use the method described in [DJZ1]. We can find a coordinate system (x_1, \dots, x_m) on $(\mathbb{C}^m, 0)$ and positive integer weights $(\lambda_1, \dots, \lambda_m)$ and quasi-homogeneous function-germs $f_1, \dots, f_k \in \mathcal{O}(\mathbb{C}^m)$ (in this coordinate systems with

these weights) such that $I = \langle f_1, \dots, f_k \rangle$. Let δ_i be a quasi-degree of f_i for $i = 1, \dots, k$.

Let F_t be a map defined in (2.1) and let V_t be a vector field along F_t for $t \in [0; 1]$ such that $V_t \circ F_t = F'_t$.

Then we have $F_0^* \omega = 0$ and it follows

$$\omega = F_1^* \omega - F_0^* \omega = \int_0^1 (F_t^* \omega)' dt = \int_0^1 F_t^* d(V_t \lrcorner \omega) dt = d \left(\int_0^1 F_t^* (V_t \lrcorner \omega) dt \right).$$

Let $\alpha = \int_0^1 F_t^* (V_t \lrcorner \omega) dt$, then $\omega = d\alpha$. But ω belongs to $I\Lambda^p(\mathbb{C}^m)$. It implies that there exist germs of p -forms β_i in $\Lambda^p(\mathbb{C}^m)$ for $i = 1, \dots, k$ such that $\omega = \sum_{i=1}^k f_i \beta_i$. So we have that

$$\alpha = \int_0^1 F_t^* (V_t \lrcorner \sum_{i=1}^k f_i \beta_i) dt = \sum_{i=1}^k f_i \int_0^1 t^{\delta_i} F_t^* (V_t \lrcorner \beta_i) dt.$$

Thus α belongs to $I\Lambda^{p-1}(\mathbb{C}^m)$. \square

The method of algebraic restrictions applied to finitely-generated quasi-homogeneous ideals is based on the following theorem.

Theorem 2.6 (a modification of Theorem A in [DJZ2]). *Let I be a finitely generated quasi-homogeneous ideal in $\mathcal{O}(\mathbb{C}^{2n})$.*

- (1) *If ω_0, ω_1 be germs at 0 of symplectic forms on \mathbb{C}^{2n} with the same algebraic restriction to I then there exists a \mathbb{C} -analytic diffeomorphism-germ Φ of \mathbb{C}^{2n} at 0 of the form $\Phi(x) = (x_1 + \phi_1(x), \dots, x_{2n} + \phi_{2n}(x))$, where $\phi_i \in I$ for $i = 1, \dots, 2n$, such that $\Phi^* \omega_1 = \omega_0$.*
- (2) *\mathbb{C} -analytic map-germs $f = (f_1, \dots, f_k), g = (g_1, \dots, g_k) : (\mathbb{C}^{2n}, 0) \rightarrow (\mathbb{C}^k, 0)$ on the symplectic space $(\mathbb{C}^{2n}, \omega)$ are symplectically V -equivalent if and only if algebraic restrictions $[\omega]_{\langle f_1, \dots, f_k \rangle}$ and $[\omega]_{\langle g_1, \dots, g_k \rangle}$ are diffeomorphic.*

Remark 2.7. It is obvious that if $\Phi(x) = (x_1 + \phi_1(x), \dots, x_{2n} + \phi_{2n}(x))$ where $\phi_i \in I$ for $i = 1, \dots, 2n$ then $\Phi^* I = I$.

A proof of Theorem 2.6 can be obtain by a small modification of the proof of Theorem A in [DJZ2]. One only needs Lemma 2.5 and the following fact.

Lemma 2.8. *Let I be a finitely generated ideal in $\mathcal{O}(\mathbb{C}^m)$. Let $X_t = \sum_{i=1}^m f_{i,t} \frac{\partial}{\partial x_i}$ for $t \in [0; 1]$ be a family of germs of \mathbb{C} -analytic vector fields on \mathbb{C}^m such that $f_{i,t} \in I$ for $i = 1, \dots, m$.*

If Φ_t for $t \in [0, 1]$ is a family of diffeomorphism-germs of $(\mathbb{C}^m, 0)$ such that

$$(2.2) \quad \frac{d}{dt} \Phi_t = X_t \circ \Phi_t$$

then

$$(2.3) \quad \Phi_t(x) = (x_1 + \phi_{1,t}(x), \dots, x_{2n} + \phi_{2n,t}(x)),$$

where $\phi_{i,t} \in I$ for $i = 1, \dots, 2n$.

A sketch of the proof. The map $t \mapsto \Phi_t(x)$ is a solution of ODE $\frac{dy}{dt} = X_t(y)$ with the initial condition $y(0) = x$. So $\Phi_t(x)$ can be obtained as a limit $\lim_{n \rightarrow \infty} T^n \Psi$ where $\Psi(t, x) \equiv x$ and $(T\Psi)(t, x) = x + \int_0^t X_s(\Psi(s, x)) ds$ is the Picard's operator.

It is easy to see that if Ψ has the form (2.3) then $T\Psi$ has the form (2.3) too. The ideal I is finitely generated. Thus Φ_t has also this form. \square

Theorem 2.6 reduces the problem of symplectic classification of quasi-homogeneous ideals to the problem of classification of the algebraic restrictions of the germ of the symplectic form to quasi-homogeneous ideals.

The meaning of the zero algebraic restriction is explained by the following theorem.

Theorem 2.9 (a modification of Theorem B in [DJZ2]). *A finitely generated quasi-homogeneous ideal I of $\mathcal{O}(\mathbb{C}^{2n})$ contains the ideal of \mathbb{C} -analytic function-germs vanishing on the germ of a non-singular Lagrangian submanifold of the symplectic space $(\mathbb{C}^{2n}, \omega)$ if and only if the symplectic form ω has zero algebraic restriction to I .*

We now formulate the modifications of basic properties of algebraic restrictions ([DJZ2]). First we can reduce the dimension of the manifold we consider due to the following propositions.

If the ideal I in $\mathcal{O}(\mathbb{C}^m)$ contains a ideal $I(M)$ of function-germs vanishing on a non-singular submanifold $M \subset \mathbb{C}^m$ then the classification of the algebraic restrictions to I of p -forms on \mathbb{C}^m reduces to the classification of the algebraic restrictions to $I|_M = \{f|_M : f \in I\}$ of p -forms on M . At first note that the algebraic restrictions $[\omega]_I$ and $[\omega|_{TM}]_{I|_M}$ can be identified:

Proposition 2.10. *Let I be an ideal in $\mathcal{O}(\mathbb{C}^m)$ which contains a ideal of function-germs vanishing on a non-singular submanifold $M \subset \mathbb{C}^m$ and let ω_1, ω_2 be germs of p -forms on \mathbb{C}^m . Then $[\omega_1]_I = [\omega_2]_I$ if and only if $[\omega_1|_{TM}]_{I|_M} = [\omega_2|_{TM}]_{I|_M}$.*

The following, less obvious statement, means that the *orbits* of the algebraic restrictions $[\omega]_N$ and $[\omega|_{TM}]_N$ also can be identified.

Proposition 2.11. *Let I_1, I_2 be ideals in the ring $\mathcal{O}(\mathbb{C}^m)$, which contain $I(M_1)$ and $I(M_2)$ respectively, where M_1, M_2 are equal-dimensional non-singular submanifolds. Let ω_1, ω_2 be two germs of p -forms. The algebraic restrictions $[\omega_1]_{I_1}$ and $[\omega_2]_{I_2}$ are diffeomorphic if and only if the algebraic restrictions $[\omega_1|_{TM_1}]_{I_1|_{M_1}}$ and $[\omega_2|_{TM_2}]_{I_2|_{M_2}}$ are diffeomorphic.*

To calculate the space of algebraic restrictions of germs of 2-forms we will use the following obvious properties.

Proposition 2.12. *If $\omega \in \mathcal{A}_0^k(I, \mathbb{C}^{2n})$ then $d\omega \in \mathcal{A}_0^{k+1}(I, \mathbb{C}^{2n})$ and $\omega \wedge \alpha \in \mathcal{A}_0^{k+p}(I, \mathbb{C}^{2n})$ for any germ of \mathbb{C} -analytic p -form α on \mathbb{C}^{2n} .*

Then we need to determine which algebraic restrictions of closed 2-forms are realizable by symplectic forms. This is possible due to the following fact.

Proposition 2.13. *Let I be an ideal of $\mathcal{O}(\mathbb{C}^{2n})$. Let r be the minimal dimension of non-singular submanifolds M of \mathbb{C}^{2n} such that I contains the ideal $I(M)$. The algebraic restriction $[\theta]_I$ of the germ of a closed 2-form θ is realizable by the germ of a symplectic form on \mathbb{C}^{2n} if and only if $\text{rank}(\theta|_{T_0M}) \geq 2r - 2n$.*

3. DISCRETE SYMPLECTIC INVARIANTS.

We use discrete symplectic invariants to distinguish symplectic singularity classes. We modify definitions of these invariants introduced in [DJZ2] for the symplectic V -equivalence.

The first invariant is a symplectic multiplicity ([DJZ2]) introduced in [IJ1] as a symplectic defect of a curve.

Let $f : (\mathbb{C}^{2n}, 0) \rightarrow (\mathbb{C}^k, 0)$ be the germ of a \mathbb{C} -analytic map on the symplectic space $(\mathbb{C}^{2n}, \omega)$.

Definition 3.1. The **symplectic multiplicity** $\mu_{\text{sympt}}(f)$ of f is the codimension of the symplectic V -orbit of f in the V -orbit of f .

The second invariant is the index of isotropy [DJZ2].

Definition 3.2. The **index of isotropy** $\iota(f)$ of $f = (f_1, \dots, f_k)$ is the maximal order of vanishing of the 2-forms $\omega|_{TM}$ over all smooth submanifolds M such that the ideal $\langle f_1, \dots, f_k \rangle$ contains $I(M)$.

These invariants can be described in terms of algebraic restrictions.

Proposition 3.3 ([DJZ2]). *The symplectic multiplicity of the germ of a quasi-homogeneous map $f = (f_1, \dots, f_k)$ on the symplectic space $(\mathbb{C}^{2n}, \omega)$ is equal to the codimension of the orbit of the algebraic restriction $[\omega]_{\langle f_1, \dots, f_k \rangle}$ with respect to the group of diffeomorphism-germs preserving the ideal $\langle f_1, \dots, f_k \rangle$ in the space of the algebraic restrictions of closed 2-forms to $\langle f_1, \dots, f_k \rangle$.*

Proposition 3.4 ([DJZ2]). *The index of isotropy of the germ of a quasi-homogeneous map $f = (f_1, \dots, f_k)$ on the symplectic space $(\mathbb{C}^{2n}, \omega)$ is equal to the maximal order of vanishing of closed 2-forms representing the algebraic restriction $[\omega]_{\langle f_1, \dots, f_k \rangle}$.*

We will use these invariants to distinguish symplectic singularities.

4. V -SIMPLE MAPS

We recall some results on classification of V -simple germs (for details see [AVG]).

Definition 4.1. The germ $f : (\mathbb{C}^m, 0) \rightarrow (\mathbb{C}^n, 0)$ is said be **V -simple** if its k -jet, for any k , has a neighborhood in the small jet space $J_{0,0}^k(\mathbb{C}^m, \mathbb{C}^n)$ that intersects only a finite number of V -equivalence classes (bounded by a constant independent of k).

Definition 4.2. The p -**parameter suspension of the map-germ** $f : (\mathbb{C}^m, 0) \rightarrow (\mathbb{C}^n, 0)$ is the map germ

$$F : (\mathbb{C}^m \times \mathbb{C}^p, 0) \ni (y, z) \mapsto (f(y), z) \in (\mathbb{C}^n \times \mathbb{C}^p, 0).$$

Theorem 4.3 (see [AVG]). *The V -simple map-germs $(\mathbb{C}^m, 0) \rightarrow (\mathbb{C}^n, 0)$ with $m \geq n$ belong, up to V -equivalence and suspension, to one of the three lists: the $A-D-E$ singularities of map-germs $\mathbb{C}^m \rightarrow \mathbb{C}$ (hypersurfaces with an isolated singularity), $S-T-U-W-Z$ singularities of map-germs $\mathbb{C}^3 \rightarrow \mathbb{C}^2$ (1-dimensional ICISs) and singularities of map-germs $\mathbb{C}^2 \rightarrow \mathbb{C}^2$ (0-dimensional ICISs) presented in Table 1.*

The normal forms in Table 1 were obtained in [G] by M. Giusti.

Notation	Normal form	Restrictions
$I_{a,b}$	$(yz, y^a + z^b)$	$a \geq b \geq 2$
I_{2a+1}	$(y^2 + z^3, z^a)$	$a \geq 3$
I_{2a+4}	$(y^2 + z^3, yz^a)$	$a \geq 2$
I_{a+5}	$(y^2 + z^a, yz^2)$	$a \geq 4$
I_{10}^*	(y^2, z^4)	-

TABLE 1. V-simple map-germs $\mathbb{C}^2 \rightarrow \mathbb{C}^2$.

5. SYMPLECTIC 0-DIMENSIONAL ICISS

We use the method of algebraic restrictions to obtain a complete classification of singularities presented in Table 1.

Theorem 5.1. *Any map-germ $(\mathbb{C}^{2n}, 0) \rightarrow (\mathbb{C}^{2n}, 0)$ from the symplectic space $(\mathbb{C}^{2n}, \sum_{i=1}^n dp_i \wedge dq_i)$ which is V-equivalent (up to a suitable suspension) to one of the normal forms in Table 1 is symplectically V-equivalent to one and only one of the following normal forms presented in Table 2*

Symplectic class	Normal forms	cod	μ_{symp}	i
$I_{a,b}^0, (n \geq 1)$	$(p_1 q_1, p_1^a + q_1^b, p_2, q_2, \dots, p_n, q_n)$	0	0	0
$I_{a,b}^1, (n \geq 2)$	$(p_1 p_2, p_1^a + p_2^b, q_1, q_2, p_3, q_3, \dots, p_n, q_n)$	1	1	∞
$I_{2a+1}^0, (n \geq 1)$	$(p_1^2 + q_1^3, q_1^a, p_2, q_2, \dots, p_n, q_n)$	0	0	0
$I_{2a+1}^1, (n \geq 2)$	$(p_1^2 + p_2^3, p_2^a, q_1, q_2 + p_1 p_2, p_3, q_3, \dots, p_n, q_n)$	1	1	1
$I_{2a+1}^2, (n \geq 2)$	$(p_1^2 + p_2^3, p_2^a, q_1, q_2, p_3, q_3, \dots, p_n, q_n)$	2	2	∞
$I_{2a+4}^0, (n \geq 1)$	$(p_1^2 + q_1^3, p_1 q_1^a, p_2, q_2, \dots, p_n, q_n)$	0	0	0
$I_{2a+4}^1, (n \geq 2)$	$(p_1^2 + p_2^3, p_1 p_2^a, q_1, q_2 + p_1 p_2, p_3, q_3, \dots, p_n, q_n)$	1	1	1
$I_{2a+4}^2, (n \geq 2)$	$(p_1^2 + p_2^3, p_1 p_2^a, q_1, q_2, p_3, q_3, \dots, p_n, q_n)$	2	2	∞
$I_{a+5}^0, (n \geq 1)$	$(p_1^2 + q_1^a, p_1 q_1^2, p_2, q_2, \dots, p_n, q_n)$	0	0	0
$I_{a+5}^1, (n \geq 2)$	$(p_1^2 + p_2^a, p_1 p_2^2, q_1, q_2 + p_1 p_2, p_3, q_3, \dots, p_n, q_n)$	1	1	1
$I_{a+5}^2, (n \geq 2)$	$(p_1^2 + p_2^a, p_1 p_2^2, q_1, q_2, p_3, q_3, \dots, p_n, q_n)$	2	2	∞
$I_{10}^{*0}, (n \geq 1)$	$(p_1^2, q_1^4, p_2, q_2, \dots, p_n, q_n)$	0	0	0
$I_{10}^{*1}, (n \geq 2)$	$(p_1^2, p_2^4, q_1, q_2 + p_1 p_2, p_3, q_3, \dots, p_n, q_n)$	1	1	1
$I_{10}^{*2}, (n \geq 2)$	$(p_1^2, p_2^4, q_1, q_2 + p_1 p_2^2, p_3, q_3, \dots, p_n, q_n)$	2	2	2
$I_{10}^{*3}, (n \geq 2)$	$(p_1^2, p_2^4, q_1, q_2, p_3, q_3, \dots, p_n, q_n)$	3	3	∞

TABLE 2. Classification of symplectic 0-dimensional isolated complete intersection singularities, cod – codimension of the classes; μ_{symp} – symplectic multiplicity; i – index of isotropy.

Proof. In the case $n = 1$ the proof follows from results in [DR] where it was proved that for quasi-homogeneous singularities in the \mathbb{C} -analytic category V-orbits coincide with volume-preserving V-orbits. For general n we present the proof in the case of I_{10}^* singularity where there are 4 different symplectic singularity classes, and in the case of I_{a+5} singularity. The proofs in other cases are very similar.

For I_{10}^* singularity we calculate the space of algebraic restrictions of 2-forms to the ideal $I = \langle y^2, z^4, x_1, \dots, x_{2n-2} \rangle$. The ideal generated by x_1, \dots, x_{2n-2} is contained in I . So by Proposition 2.10 we may consider the following ideal $J = I|_{\{x_1 = \dots = x_{2n-2} = 0\}} = \langle y^2, z^4 \rangle$ in the ring $\mathcal{O}(\mathbb{C}^2)$. By Proposition 2.12 germs

of 1-forms $d(1/2y^2) = ydy$, $d(1/4z^4) = z^3dz$ and germs of 2-forms $ydy \wedge dz$, $z^3dy \wedge dz$ have zero algebraic restriction to J . So any algebraic restriction of the germ of a closed 2-forms to J can be presented in the following form $[\omega]_J = A[dy \wedge dz]_J + B[zdy \wedge dz]_J + C[z^2dy \wedge dz]_J$, where $A, B, C \in \mathbb{C}$.

If $A \neq 0$ then we obtain $\Phi^*[\omega]_J = [dy \wedge dz]_J$ by the diffeomorphism-germ of the form $\Phi(y, z) = (y, z(A + 1/2Bz + 1/3Cz^2))$. If $A = 0$ and $B \neq 0$ then we obtain $\Phi^*[\omega]_J = [zdy \wedge dz]_J$ by the diffeomorphism-germ of the form $\Phi(y, z) = (y, z\phi(z))$, where $\phi^2(z) = B + 2/3Cz$. If $A = B = 0$ and $C \neq 0$ then we obtain $\Phi^*[\omega]_J = [z^2dy \wedge dz]_J$ by the diffeomorphism-germ of the form $\Phi(y, z) = (Cy, z)$.

Since the minimal dimension r of the germ of a non-singular submanifold M such that $I(M) \subset I$ is 2 then by Proposition 2.13 for $n = 1$ only the algebraic restriction $[dy \wedge dz]_I$ is realizable by the germ of a symplectic form.

For $n > 1$ all algebraic restrictions are realizable by the following symplectic forms:

$$(5.1) \quad dy \wedge dz + \sum_{i=1}^{n-1} dx_{2i-1} \wedge dx_{2i},$$

$$(5.2) \quad zdy \wedge dz + dy \wedge dx_1 + dz \wedge dx_2 + \sum_{i=2}^{n-1} dx_{2i-1} \wedge dx_{2i},$$

$$(5.3) \quad z^2dy \wedge dz + dy \wedge dx_1 + dz \wedge dx_2 + \sum_{i=2}^{n-1} dx_{2i-1} \wedge dx_{2i},$$

$$(5.4) \quad dy \wedge dx_1 + dz \wedge dx_2 + \sum_{i=2}^{n-1} dx_{2i-1} \wedge dx_{2i}.$$

By a simple change of coordinates we obtain the normal forms in Table 2.

For I_{a+5} singularity the space algebraic restrictions of germs of closed 2-forms to the ideal $I = \langle y^2 + z^a, yz^2, x_1, \dots, x_{2n-2} \rangle$ can be calculated in the same way. We obtain that any algebraic restriction of the germs of a closed 2-forms on $\mathbb{C}^2 = \{x_1 = \dots = x_{2n-2} = 0\}$ to $J = I|_{\{x_1 = \dots = x_{2n-2} = 0\}} = \langle y^2 + z^a, yz^2 \rangle$ can be presented in the following form

$$(5.5) \quad [\omega]_J = A[dy \wedge dz]_J + B[zdy \wedge dz]_J,$$

where $A, B \in \mathbb{C}$.

First assume that $A \neq 0$. Let E denote the germ of the Euler vector field $ay\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z}$. Then it is easy to check that a flow Φ_t of the germ of a vector field $X = \frac{B}{(a+4)A}zE$ preserves J , $\mathcal{L}_X(Ady \wedge dz) = Bzdy \wedge dz$, $[\mathcal{L}_X(Bzdy \wedge dz)]_J = 0$. Therefore $\Phi_t^*[Ady \wedge dz + tBzdy \wedge dz]_J = [Ady \wedge dz]_J$ for $t \in [0; 1]$ (see [D]). Finally by a linear change of coordinates of the form $(y, z) \mapsto (Cy, Dz)$, where for $C, D \in \mathbb{C}$ such that $C^2 = D^a$ and $CD = A$ we show that if $A \neq 0$ then the algebraic restriction (5.5) is diffeomorphic to $[dy \wedge dz]_J$. By a similar change of coordinates preserving J we show that if $A = 0$ and $B \neq 0$ then the algebraic restriction (5.5) is diffeomorphic to $[zdy \wedge dz]_J$. As in the previous case, for $n = 1$ only $[dy \wedge dz]_I$ can be realizable by the germ of a symplectic form. For $n \geq 2$ algebraic restrictions are realizable by (5.1), (5.2) and (5.4). Normal forms in Table 2 are obtained by an obvious change of coordinates. \square

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